

Excess information in Parametric Linear Optimization*

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We consider a parametric linear optimization problem (called primal) and its corresponding dual problem, where the parameters are the cost vector and the right-hand-side vector, respectively. This paper characterizes those constraints of the primal problem (variables of the dual problem, respectively) which can be eliminated without modifying its feasible set mapping, its optimal set mapping, and its value mapping. Superfluity relative to the primal feasible set is nothing else than redundancy in its constraint system, whereas superfluity relative to the dual optimal set is closely related with another well-known phenomenon of excess of information in linear optimization: strong strangeness. The relationships between all these phenomena are also analyzed.

Key Words: linear inequality systems, linear programming, linear semi-infinite programming, excess of information, redundancy.

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1 Introduction

We consider given a linear system in \mathbb{R}^n , $\sigma = \{a'_t x \geq b_t, t \in T\}$, where T is an arbitrary index set with cardinality $2 \leq |T| \leq \infty$. We associate with σ the parametric linear optimization problem

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$$P(c) : \inf c'x \quad \text{s.t.} \quad a'_t x \geq b_t, \quad t \in T,$$

and its dual

$$D(c) : \sup \Psi(\lambda) := \sum_{t \in T} \lambda_t b_t \quad \text{s.t.} \quad \sum_{t \in T} \lambda_t a_t = c, \quad \lambda \in \mathbb{R}_+^{(T)},$$

where the parameter is $c \in \mathbb{R}^n$ whereas $X^{(T)}$, with $0 \in X \subset \mathbb{R}$, denotes the set of the mappings $\lambda : T \rightarrow X$ vanishing everywhere except on a finite subset of T , $\text{supp } \lambda = \{t \in T \mid \lambda_t \neq 0\}$. We denote by F , $F^*(c)$ and $v(c)$ the feasible set, the optimal set and the optimal value of $P(c)$, and we denote by $\Lambda(c)$, $\Lambda^*(c)$ and $v^D(c)$ the feasible set, the optimal set and the optimal value of $D(c)$, respectively. By definition, $v(c) = +\infty$ if $F = \emptyset$ and $v^D(c) = -\infty$ if $\Lambda(c) = \emptyset$.

If $|T| < \infty$, $P(c)$ is a linear programming (LP) problem in canonical form and $D(c)$ is a LP problem in standard form. In this case, if at least one of the two problems is bounded, then they are solvable and $v(c) = v^D(c)$.

If $|T| = \infty$, $P(c)$ is a primal linear semi-infinite (LSIP) problem and $D(c)$ is its Haar's dual problem. It is possible that $v^D(c) < v(c)$ even though both problems (possibly unsolvable) are consistent.

We consider the feasible set of $P(c)$, F (which can be seen as a constant set-valued mapping), the set-valued mappings $F^* : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $\Lambda, \Lambda^* : \mathbb{R}^n \rightrightarrows \mathbb{R}^{(T)}$, and the ordinary mappings $v, v^D : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$. Similarly, given an index $s \in T$, we associate with the *relaxed system* $\sigma_s := \{a'_t x \geq b_t, t \in T \setminus \{s\}\}$ its corresponding parametric problems, $P_s(c)$ and $D_s(c)$, and their corresponding mappings: $F_s, F_s^*, \Lambda_s, \Lambda_s^*, v_s$ and v_s^D .

We say that the constraint $a'_s x \geq b_s$ (the variable λ_s) is *superfluous* relative to one of the mappings associated with $P(c)$ ($D(c)$, respectively) if its elimination does not modify the corresponding mapping. For the sake of brevity we say that $s \in T$ is *superfluous* in both cases. In particular, s is *superfluous relative to the primal feasible set* if $F_s = F$ (i.e., s is redundant), s is *superfluous relative to the primal value function* (PVS) if $v_s = v$, and s is *superfluous relative to the primal optimal set* (POS) if $F_s^* = F^*$. Similarly, we say that s is *superfluous relative to the dual optimal value* (DVS) if $v_s^D = v^D$.

Concerning the dual set-valued mappings, since the decision spaces of $D(c)$ and $D_s(c)$ are $\mathbb{R}^{(T)}$ and $\mathbb{R}^{(T \setminus \{s\})}$, respectively, the comparison of subsets requires the identification of each subset of $\mathbb{R}^{(T \setminus \{s\})}$ with another one in $\mathbb{R}^{(T)}$. The natural way to do this consists of associating with each $\gamma \in \mathbb{R}^{(T \setminus \{s\})}$ its extension with zero $\bar{\gamma} \in \mathbb{R}^{(T)}$, i.e.,

$$\bar{\gamma}_t = \begin{cases} \gamma_t, & t \in T \setminus \{s\}, \\ 0, & t = s, \end{cases}$$

so that we identify any set $\Gamma \subset \mathbb{R}^{(T \setminus \{s\})}$ with the set $\bar{\Gamma} := \{\bar{\gamma} \in \mathbb{R}^{(T)} \mid \gamma \in \Gamma\}$. In particular, we have $\overline{\Lambda_s(c)} \subset \Lambda(c)$ for all $c \in \mathbb{R}^n$, but, in contrast with the primal feasible set (where $F_s = F$ is possible) we always have $\overline{\Lambda_s} \neq \Lambda$. In fact, defining $\lambda \in \mathbb{R}_+^{(T)}$ such that

$$\lambda_t = \begin{cases} 1, & t = s, \\ 0, & t \in T \setminus \{s\}, \end{cases}$$

we have $\lambda \in \Lambda(a_s)$, so that $\Lambda(a_s) \neq \bar{\Gamma}$ for all $\Gamma \subset \mathbb{R}_+^{(T \setminus \{s\})}$. So, no constraint in σ is superfluous relative to the dual feasible set.

Appealing to this notation, we say that s is *superfluous relative to the dual optimal set* (DOS) if $\overline{\Lambda_s^*} = \Lambda^*$.

The superfluous constraints and variables are related with other types of unnecessary information. For instance, we say that $s \in T$ is *nonessential* if its corresponding constraint $a'_s x \geq b_s$ is not binding at the set of extreme points of $F^*(c)$ for all $c \in \mathbb{R}^n$. On the other hand, a variable is said to be extraneous (strongly extraneous) in a standard LP problem

$$\text{Max } c'x \quad \text{s.t.} \quad Ax = b, x \geq 0_n,$$

where A and c are fixed and b is the parameter, if this variable vanishes at some (all) optimal solution for any vector b such that the above LP problem is solvable. Translating these concepts to our general context (T arbitrary), we say that $s \in T$ is *extraneous* (*strongly extraneous*) if for every $c \in \mathbb{R}^n$, either $\Lambda^*(c) = \emptyset$ or $s \notin \text{supp } \lambda$ for some (all, respectively) $\lambda \in \Lambda^*(c)$.

The paper is mainly intended to provide tests for checking each of the eight superfluity phenomena in parametric linear optimization we have just defined. These geometric characterizations are given in Sections from 3 to 6. From them, we obtain in Section 7 a diagram (Figure 1) showing all the connections between all these phenomena.

The next example shows that all these phenomena not only are possible but may occur simultaneously.

Example 1.1 Let $\sigma = \{x_1 \geq 1; x_1 \geq -1\}$ in \mathbb{R}^2 . It is easy to see that $s = 2$ is redundant, POS, PVS and nonessential. In order to show that it is also DOS, DVS and strongly extraneous we shall discuss the position of c relative to the so-called *first moment cone* of σ , $M := \text{cone}\{a_t, t \in T\}$. If $c \notin M$, $D(c)$ and $D_s(c)$ are inconsistent, so that $\Lambda^*(c) = \Lambda_s^*(c) = \emptyset$ and $v^D(c) = v_s^D(c) = -\infty$. If $c \in M$, we can write $c = \mu(1, 0)'$ with $\mu \geq 0$. We have $\Lambda_2(c) = \{\mu\}$ and $\Lambda_2^*(c) = \{\mu\}$, with $v_2^D(c) = \mu$. On the other hand, $\Lambda(c) = \{(\lambda_1, \lambda_2)' \in \mathbb{R}_+^2 \mid \lambda_1 + \lambda_2 = \mu\}$ and $\Lambda^*(c) = \{(\mu, 0)'\}$, with $v^D(c) = \mu$. Hence, $\overline{\Lambda_2^*(c)} = \Lambda^*(c)$ and $v_2^D(c) = v^D(c)$ for all $c \in \mathbb{R}^2$. Consequently, $s = 2$ is DOS and DVS. Moreover, if $\Lambda^*(c) \neq \emptyset$, then it is a singleton set and the second coordinate of its unique element is zero. Thus, s is strongly extraneous.

The first papers dealing with redundancy are due to Boot [4] and Charnes, Cooper and Thompson [5]. Since then many works have been written on this phenomenon (see, e.g., [10], [3], [7], and references therein). The extraneous variables were introduced in [5] and the strongly extraneous variables in [11] (see also [2], [6], and references therein). With the only exception of redundancy, fixing c we get less restrictive concepts (i.e., excess of information phenomena in nonparametric linear optimization). For instance, Mauri [12] considered extraneous variables in LP whereas Goberna, Jornet and Molina [9] analyzed PVS and POS constraints in LP and LSIP.

Generally speaking, the existence of an excess of information in an optimization problem affects its theoretical properties and the computational efficiency of the numerical methods. Aardal [1] and Zhu and Broughan [13] have identified optimization problems in which the aggregation or the elimination of superfluous information provides important benefits. Concerning linear optimization, in LP the unfavorable effects of the excess of information outnumber the favorable ones ([11]), whereas the situation is the opposite in LSIP ([9]).

2 Preliminaries

Let us introduce the necessary notation and basic results (whose proofs can be found in [8]).

Given a set $\emptyset \neq X \subset \mathbb{R}^n$, we denote by $\text{cone } X$, $\text{span } X$, and $\text{conv } X$ the convex

cone spanned by X , the linear span of X and the convex hull of X , respectively. From the topological side, $\text{cl } X$ denote the closure of X and $\text{bd } X$ its boundary. If $X \neq \emptyset$ is a convex subset of a linear space, $\text{extr } X$ and O^+X denote its set of extreme points and its recession cone, respectively. By definition, $\text{extr } \emptyset = \emptyset$.

We associate with σ its *first-moment cone*, M , defined in Example 1.1 and its *characteristic cone*,

$$K := \text{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}.$$

σ is consistent if and only if $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \notin \text{cl } K$. Moreover, σ is *strongly inconsistent* (i.e., σ contains at least a finite inconsistent subsystem) if and only if $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in K$.

We associate with $s \in T$ the characteristic and the first moment cone of σ_s , denoted by K_s and M_s , and two intervals, J_s and I_s , defined as

$$J_s := \left\{ \alpha \in \mathbb{R} \mid \begin{pmatrix} a_s \\ \alpha \end{pmatrix} \in K_s \right\} \subset I_s := \left\{ \alpha \in \mathbb{R} \mid \begin{pmatrix} a_s \\ \alpha \end{pmatrix} \in K \right\},$$

which can be empty, (open or closed) halflines or the whole real line \mathbb{R} . Obviously, $I_s \neq \emptyset$ whereas $J_s \neq \emptyset$ if and only if $a_s \in M_s$. Moreover, $\max \{b_s, \sup J_s\} \leq \sup I_s \leq +\infty$. It is easy to see that $\sup I_s = +\infty$ entails the inconsistency of σ and, conversely, if σ is strongly inconsistent, then $\sup I_s = +\infty$. Similarly, $\sup J_s = +\infty$ entails the inconsistency of σ_s and, conversely, if σ_s is strongly inconsistent and $a_s \in M_s$, then $\sup J_s = +\infty$.

Let $H_s := \{x \in \mathbb{R}^n \mid a'_s x = b_s\}$ (a hyperplane if $a_s \neq 0_n$). Then $\sigma_s \cup \{a'_s x = b_s\}$ is a linear representation of $F \cap H_s$. If σ is consistent and s (or $a'_s x \geq b_s$) is *nonbinding* (i.e., $F \cap H_s = \emptyset$), then s is redundant. In particular, if $\sigma_s \cup \{a'_s x = b_s\}$ is strongly inconsistent, then s is called *strongly redundant*. If σ is consistent, s is strongly redundant if and only if there exists some $\varepsilon > 0$ such that $\begin{pmatrix} a_s \\ b_s + \varepsilon \end{pmatrix} \in K_s$.

Lemma 2.1 *If σ is not strongly inconsistent and $J_s \neq I_s$, then $\max I_s = b_s$.*

Proof. We suppose that σ is not strongly inconsistent and $I_s \setminus J_s \neq \emptyset$. If $\alpha \in J_s$ and $\beta \in I_s \setminus J_s$, then $\alpha < \beta$.

Let $\beta \in I_s \setminus J_s$. Since $\begin{pmatrix} a_s \\ \beta \end{pmatrix} \in K \setminus K_s$, we can write

$$\begin{pmatrix} a_s \\ \beta \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \delta \begin{pmatrix} a_s \\ b_s \end{pmatrix}, \quad (1)$$

with $\begin{pmatrix} a \\ b \end{pmatrix} \in K_s$ and $\delta > 0$. Two cases can arise for δ .

Case 1: $\delta < 1$. From (1) we get

$$\frac{1}{1-\delta} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{1-\delta} \begin{pmatrix} a_s(1-\delta) \\ \beta - \delta b_s \end{pmatrix} = \begin{pmatrix} a_s \\ \frac{\beta - \delta b_s}{(1-\delta)} \end{pmatrix} \in K_s.$$

Then, $\frac{\beta - \delta b_s}{(1-\delta)} \leq \sup J_s < b_s$. Thus, $\beta - \delta b_s < b_s - \delta b_s$, i.e., $\beta < b_s$.

Case 2: $\delta \geq 1$. Again from (1) we get

$$\begin{pmatrix} a_s(1-\delta) \\ \beta - \delta b_s \end{pmatrix} + (\delta - 1) \begin{pmatrix} a_s \\ b_s \end{pmatrix} = (\beta - b_s) \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in K,$$

and this entails, since σ is not strongly inconsistent, that $\beta - b_s \leq 0$, i.e., $\beta \leq b_s$.

We have shown that $\beta \leq b_s$ for all $\beta \in I_s \setminus J_s$, so that $\sup I_s \leq b_s$. Since $b_s \in I_s$ we conclude that $\max I_s = b_s$. ■

Now assume that σ is consistent. An inequality $a'x \geq b$ is consequence of σ if and only if $\begin{pmatrix} a \\ b \end{pmatrix} \in \text{cl } K$ (Farkas' Lemma). So, K is closed if and only if every consequence of σ is also the consequence of a finite subsystem. In this case σ is said to be *Farkas-Minkowsy* (FM). If σ is an ordinary system (i.e., $|T| < \infty$), then it is FM.

With respect to the dual problem $D(c)$, we have

$$v^D(c) = \sup \left\{ \alpha \in \mathbb{R} \mid \begin{pmatrix} c \\ \alpha \end{pmatrix} \in K \right\} \quad (2)$$

(and this is different of $-\infty$ if and only if $c \in M$).

3 Superfluous constraints relative to the primal mappings

Proposition 3.1 *Given $s \in T$, the following statements are equivalent to each other:*

- (i) s is redundant.
- (ii) s is PVS.

(iii) s is POS.

(iv) $\left\{ \begin{pmatrix} a_s \\ b_s \end{pmatrix}, \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\} \cap \text{cl } K_s \neq \emptyset$.

Proof. If $a_t = 0_n$ for all $t \in T$, it is easy to prove that the statements (i)-(iv) only fail (simultaneously) when $b_t \leq 0$ for all $t \in T \setminus \{s\}$ and $b_s > 0$ (consider $c = 0_n$). So we can assume that $\{a_t, t \in T\} \neq \{0_n\}$.

First we show that (i) \Leftrightarrow (ii) \Leftrightarrow (iii) discussing three possible cases.

Case 1: σ and σ_s are consistent.

(i) \Rightarrow (iii) \Rightarrow (ii) are trivial. In order to prove that (ii) \Rightarrow (i) we assume that (i) fails. let $\bar{x} \in F_s \setminus F$, i.e., $a'_t \bar{x} \geq b_t$, for all $t \in T \setminus \{s\}$ and $a'_s \bar{x} < b_s$. Then $v_s(a_s) \leq a'_s \bar{x} < b_s \leq v(a_s)$, so that $v_s \neq v$, i.e., (ii) fails.

Case 2: σ and σ_s are inconsistent.

Since $F = F_s = \emptyset$, $F^*(c) = F_s^*(c) = \emptyset$ for all $c \in \mathbb{R}^n$ so that statements (i)-(iii) hold.

Case 3: σ is inconsistent and σ_s is consistent.

In this case s is obviously nonredundant and we shall prove that (ii) and (iii) fail too. In fact, we have $F = \emptyset$, $F^*(c) = \emptyset$, and $v(c) = +\infty$ for all $c \in \mathbb{R}^n$. Moreover, the additional assumption guarantees that $\emptyset \neq F_s \neq \mathbb{R}^n$. Hence, there exists $\bar{x} \in \text{bd } F_s$. Let $a'x \geq b$ be a supporting halfspace for F_s at \bar{x} . Then, $\bar{x} \in F_s^*(a)$. Since $F^*(a) = \emptyset \neq F_s^*(a)$ and $v_s(a) = b < +\infty = v(a)$, (ii) and (iii) fail.

Finally, let us observe that (i) holds if and only if either $F_s = \emptyset$ (in which case $F = \emptyset$ too) or $F_s \neq \emptyset$ and $a'_s x \geq b_s$ for all $x \in F_s$, i.e., either $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \text{cl } K_s$ or $\begin{pmatrix} a_s \\ b_s \end{pmatrix} \in \text{cl } K_s$ (Farkas' Lemma), i.e., (iv) holds. \blacksquare

If s is superfluous relative to the primal mappings and σ_s is consistent, then, from (iv), $a_s \in \text{cl } M_s$. Nevertheless $a_s \in M_s$ could fail as the next example shows.

Example 3.1 Let $\sigma = \{(1-t)x_1 + tx_2 \geq 0, t \in [0, 1]\}$ and $s = 1$. It is easy to see that $F_1 = F = \mathbb{R}_+^2$, so that $s = 1$ is redundant. Nevertheless $a_1 = (0, 1)' \notin M_1$.

4 Superfluous variables relative to the dual value

Proposition 4.1 Given $s \in T$, the following statements are equivalent to each other:
(i) s is DVS.

- (ii) $v_s^D(a_s) = v^D(a_s) \neq -\infty$.
 (iii) $a_s \in M_s$ and $\sup J_s = \sup I_s$.

Proof. (i) \Rightarrow (ii) From (i) we get $v_s^D(a_s) = v^D(a_s)$. Moreover, taking $\lambda \in \mathbb{R}_+^{(T)}$ such that $\lambda_t = 1$ if $t = s$ and $\lambda_t = 0$ otherwise, we have $\lambda \in \Lambda(a_s)$. Then $v^D(a_s) \geq \Psi(\lambda) = b_s > -\infty$.

(ii) \Rightarrow (iii) Assume that $v_s^D(a_s) = v^D(a_s) \neq -\infty$. Since $D_s(a_s)$ is consistent, $a_s \in M_s$. Moreover, by (2),

$$\sup J_s = v_s^D(a_s) = v^D(a_s) = \sup I_s.$$

(iii) \Rightarrow (i) We assume that (iii) holds. Since $a_s \in M_s$, $M = M_s$.

Given $c \in \mathbb{R}^n$, if $c \notin M = M_s$, we have $v^D(c) = v_s^D(c) = -\infty$. Hence, we take $c \in M$.

We have to prove that $v^D(c) \leq v_s^D(c)$, i.e., that $\alpha < v^D(c)$ entails $\alpha < v_s^D(c)$.

Given $\alpha < v^D(c)$, there exists $\lambda \in \mathbb{R}_+^{(T)}$ and $\beta > \alpha$ such that

$$\sum_{t \in T} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} = \sum_{t \in T \setminus \{s\}} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \lambda_s \begin{pmatrix} a_s \\ b_s \end{pmatrix} = \begin{pmatrix} c \\ \beta \end{pmatrix}. \quad (3)$$

If $\lambda_s = 0$, we have $\alpha < \beta \leq v_s^D(c)$. So we assume $\lambda_s > 0$.

Since $\begin{pmatrix} a_s \\ b_s \end{pmatrix} \in K$, we have $b_s \leq \sup I_s = \sup J_s$.

Given $\varepsilon > 0$, arbitrarily small, there exists $\gamma \in \mathbb{R}_+^{(T \setminus \{s\})}$, $\eta \geq 0$ and $\delta \in \mathbb{R}$ such that $b_s - \frac{\varepsilon}{\lambda_s} < \delta$ and

$$\sum_{t \in T \setminus \{s\}} \gamma_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \begin{pmatrix} 0_n \\ -\eta \end{pmatrix} = \begin{pmatrix} a_s \\ \delta \end{pmatrix}.$$

Then,

$$\begin{pmatrix} a_s \\ b_s \end{pmatrix} = \begin{pmatrix} a_s \\ \delta \end{pmatrix} + \begin{pmatrix} 0_n \\ b_s - \delta \end{pmatrix} = \sum_{t \in T \setminus \{s\}} \gamma_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \begin{pmatrix} 0_n \\ b_s - \delta - \eta \end{pmatrix}. \quad (4)$$

Combining (3) and (4) we obtain

$$\begin{pmatrix} c \\ \beta + \lambda_s (\delta - b_s + \eta) \end{pmatrix} = \sum_{t \in T \setminus \{s\}} (\lambda_t + \lambda_s \gamma_t) \begin{pmatrix} a_t \\ b_t \end{pmatrix} \in K_s,$$

so that

$$v_s^D(c) \geq \beta + \lambda_s (\delta - b_s + \eta) > \beta - \varepsilon > \beta > \alpha.$$

This completes the proof. ■

In particular, if $K_s = K$ (e.g., if $a'_s x \geq b_s$ is repeated), then s is DVS.

Corollary 4.1 *If s is DVS, then $a_s \in M_s$. The converse statement holds if σ_s is strongly inconsistent.*

Proof. The direct statement is straightforward consequence of Proposition 4.1.

Now we assume that $a_s \in M_s$ and σ_s is strongly inconsistent. Take an arbitrary $\alpha \in J_s$. Then for all $\delta \geq 0$ we have

$$\begin{pmatrix} a_s \\ \alpha + \delta \end{pmatrix} = \begin{pmatrix} a_s \\ \alpha \end{pmatrix} + \delta \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in K_s + \text{cone} \left\{ \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\} = K_s,$$

so that $\alpha + \delta \in J_s$. Hence $+\infty = \sup J_s \leq \sup I_s$ and Proposition 4.1 applies again. ■

Proposition 4.2 *If s is DVS, then s is redundant.*

Proof. Assume that s is nonredundant. Then, according to Proposition 3.1, we have $\begin{pmatrix} a_s \\ b_s \end{pmatrix} \notin \text{cl } K_s$. We shall prove that $\beta := \sup J_s < b_s$. In fact, if $\beta \geq b_s$, then there exists a nondecreasing sequence of scalars, $\{\beta_r\}_{r \in \mathbb{N}}$, such that $\begin{pmatrix} a_s \\ \beta_r \end{pmatrix} \in K_s$ for all $r \in \mathbb{N}$ and $\lim_r \beta_r = \beta \geq b_s$. Then,

$$\begin{pmatrix} a_s \\ b_s \end{pmatrix} = \lim_r \begin{pmatrix} a_s \\ \beta_r \end{pmatrix} + \begin{pmatrix} 0_n \\ b_s - \beta \end{pmatrix} \in \text{cl } K_s,$$

in contradiction with the assumption. Hence, by (2), $v_s^D(a_s) = \beta < b_s \leq v^D(a_s)$, and s cannot be DVS. ■

The converse statement of Proposition 4.2 is not true, as Example 3.1 shows (recall that $a_1 \notin M_1$).

5 Extraneous variables and nonessential constraints

Proposition 5.1 *Given $s \in T$, the following statements are equivalent to each other:*

- (i) s is extraneous.
- (ii) If $D(a_s)$ is solvable, then $D_s(a_s)$ is solvable and $v_s^D(a_s) = v^D(a_s)$.
- (iii) If there exists $\max I_s \in \mathbb{R}$, then $I_s = J_s$.

Proof. (i) \Rightarrow (ii) We assume that (i) holds and $D(a_s)$ is solvable.

Since s is extraneous and $\Lambda^*(a_s) \neq \emptyset$, there exists $\lambda^* \in \Lambda^*(a_s)$ such that $s \notin \text{supp } \lambda^*$. Then the restriction of λ^* to $T \setminus \{s\}$ is a feasible solution of $D_s(a_s)$ such that the value of the objective functional is $\Psi(\lambda^*) = v^D(a_s) \geq v_s^D(a_s)$. Hence, $D_s(a_s)$ is solvable and $v_s^D(a_s) = v^D(a_s)$.

(ii) \Rightarrow (iii) We assume that (ii) holds and $\bar{\alpha} = \max I_s \in \mathbb{R}$. Then, recalling (2), $D(a_s)$ is solvable and $v^D(a_s) = \bar{\alpha}$. Moreover, $\sup J_s = v_s^D(a_s) = v^D(a_s) = \bar{\alpha}$ and, due to the solvability of $D_s(a_s)$, we have $\max I_s = \bar{\alpha} = \max J_s$ and so $I_s = J_s =]-\infty, \bar{\alpha}]$.

(iii) \Rightarrow (i) We assume (iii). Let $c \in M$ such that $\Lambda^*(c) \neq \emptyset$. Then, there exists $\gamma \in \mathbb{R}_+^{(T)}$ such that

$$c = \sum_{t \in T} \gamma_t a_t \quad \text{and} \quad v^D(c) = \sum_{t \in T} \gamma_t b_t.$$

If $\gamma_s = 0$ we have finished. So, we assume that $\gamma_s > 0$. We shall obtain another optimal solution of $D(c)$ which vanishes at s .

First we prove that $v^D(a_s) = b_s$. Since $v^D(a_s) \geq b_s$ we shall assume that $v^D(a_s) > b_s$ and we shall get a contradiction. Let $\mu \in \mathbb{R}_+^{(T)}$ such that $\sum_{t \in T} \mu_t a_t = a_s$ and $b_s < \sum_{t \in T} \mu_t b_t$. Defining $\lambda \in \mathbb{R}_+^{(T)}$ as

$$\lambda_t := \begin{cases} \gamma_t + \gamma_s \mu_t, & \text{if } t \neq s, \\ \gamma_s \mu_s, & \text{if } t = s, \end{cases}$$

we have

$$\begin{aligned} \sum_{t \in T} \lambda_t a_t &= \sum_{t \in T \setminus \{s\}} (\gamma_t + \gamma_s \mu_t) a_t + \gamma_s \mu_s a_s \\ &= \sum_{t \in T \setminus \{s\}} \gamma_t a_t + \gamma_s \sum_{t \in T} \mu_t a_t = \sum_{t \in T \setminus \{s\}} \gamma_t a_t + \gamma_s a_s = c, \end{aligned}$$

so that $\lambda \in \Lambda(c)$. Similarly,

$$\begin{aligned} \Psi(\lambda) &= \sum_{t \in T} \lambda_t b_t = \sum_{t \in T \setminus \{s\}} (\gamma_t + \gamma_s \mu_t) b_t + \gamma_s \mu_s b_s \\ &= \sum_{t \in T \setminus \{s\}} \gamma_t b_t + \gamma_s \sum_{t \in T} \mu_t b_t > \sum_{t \in T \setminus \{s\}} \gamma_t b_t + \gamma_s b_s = v^D(c), \end{aligned}$$

and this is a contradiction.

By (2), we have $b_s = v^D(a_s) = \max I_s$ and, recalling (iii), we get $b_s \in I_s = J_s$. Thus

there exists $\rho \in \mathbb{R}_+^{(T \setminus \{s\})}$ and $\xi \geq 0$ such that

$$\begin{pmatrix} a_s \\ b_s \end{pmatrix} = \sum_{t \in T \setminus \{s\}} \rho_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \xi \begin{pmatrix} 0_n \\ -1 \end{pmatrix}.$$

Finally we shall prove that $\eta \in \mathbb{R}_+^{(T)}$, defined as

$$\eta_t := \begin{cases} \gamma_t + \gamma_s \rho_t, & \text{if } t \neq s, \\ 0, & \text{if } t = s, \end{cases}$$

satisfies $\eta \in \Lambda^*(c)$ (observe that $s \notin \text{supp } \eta$). In fact,

$$\begin{aligned} \sum_{t \in T} \eta_t a_t &= \sum_{t \in T \setminus \{s\}} (\gamma_t + \gamma_s \rho_t) a_t \\ &= \sum_{t \in T \setminus \{s\}} \gamma_t a_t + \gamma_s \sum_{t \in T \setminus \{s\}} \rho_t a_t = \sum_{t \in T \setminus \{s\}} \gamma_t a_t + \gamma_s a_s = c, \end{aligned}$$

so that, $\eta \in \Lambda(c)$. On the other hand,

$$\begin{aligned} \Psi(\eta) &= \sum_{t \in T} \eta_t b_t = \sum_{t \in T \setminus \{s\}} (\gamma_t + \gamma_s \rho_t) b_t \\ &= \sum_{t \in T \setminus \{s\}} \gamma_t b_t + \gamma_s \sum_{t \in T \setminus \{s\}} \rho_t b_t = \Psi(\gamma) - \gamma_s b_s + \gamma_s (b_s + \xi) \geq v^D(c), \end{aligned}$$

so that $\eta \in \Lambda^*(c)$. The proof is complete. ■

Corollary 5.1 *If $\begin{pmatrix} a_s \\ b_s \end{pmatrix} \in K_s$, then s is extraneous. This occurs if s is redundant and σ_s is FM.*

Proof. If $\begin{pmatrix} a_s \\ b_s \end{pmatrix} \in K_s$, then $K = K_s$ and $I_s = J_s$. Then statement (iii) in Proposition 5.1 trivially holds.

Now we assume that s is redundant and σ_s is FM. Then $a'_s x \geq b_s$ is a consequence of σ_s and Farkas' Lemma yields $\begin{pmatrix} a_s \\ b_s \end{pmatrix} \in \text{cl } K_s = K_s$. ■

Example 3.1 shows that the FM assumption cannot be removed in Corollary 5.1. In fact, $s = 1$ is redundant (but it is not DVS: $v_1^D(a_1) = -\infty < 0 = v^D(a_1)$). Nevertheless, defining λ^* such that $\lambda_t^* = 1$ if $t = 1$ and $\lambda_t^* = 0$ otherwise, it is easy to see that $\Lambda(a_1) = \{\lambda^*\}$, so that $\Lambda^*(a_1) = \{\lambda^*\}$. Since $1 \in \text{supp } \lambda^*$, $s = 1$ is not extraneous.

Corollary 5.2 *If s is extraneous and σ is not strongly inconsistent, then $\sup J_s \geq b_s$.*

Proof. We suppose that $\sup J_s < b_s$. Since $b_s \leq \sup I_s$, $I_s \setminus J_s \neq \emptyset$. By Lemma 2.1, we have $\max I_s = b_s$. Hence, $I_s = J_s$ by Proposition 5.1, in contradiction with $\sup J_s < b_s$. ■

Proposition 5.2 *If σ is strongly inconsistent then any $s \in T$ is strongly extraneous. Otherwise, s is strongly extraneous if and only if $\sup J_s > b_s$. In the last case, if σ is consistent, then s is strongly redundant.*

Proof. First we assume that σ is strongly inconsistent, i.e., $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in K$. Let $s \in T$ arbitrary and let $\gamma \in \mathbb{R}_+^{(T)}$ such that

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} = \sum_{t \in T} \gamma_t \begin{pmatrix} a_t \\ b_t \end{pmatrix}.$$

Then, if $\Lambda(c) \neq \emptyset$ (i.e., $c \in M$), $\gamma \in O^+ \Lambda(c)$ and $\Psi(\gamma) = \sum_{t \in T} \gamma_t b_t = 1$. In such case $D(c)$ is unbounded (and so $\Lambda^*(c) = \emptyset$) for every $c \in \mathbb{R}^n$ and s turns out to be strongly extraneous.

Now we assume that σ is not strongly inconsistent.

Suppose that s is strongly extraneous. By Corollary 5.2, we know that $\sup J_s \geq b_s$. We shall assume that $\sup J_s = b_s$ and we shall obtain a contradiction. If $\sup I_s > b_s$, then $J_s \subsetneq I_s$. Then, by Lemma 2.1, $\max I_s = b_s$ and so $v^D(a_s) = b_s$.

Consider $\lambda \in \mathbb{R}_+^{(T)}$ such that $\lambda_t = 1$ if $t = s$ and $\lambda_t = 0$ otherwise. It can be easily realized that $\lambda \in \Lambda(a_s)$ with $\Psi(\lambda) = b_s = v^D(a_s)$. Then $\lambda \in \Lambda^*(a_s)$ and $s \in \text{supp } \lambda$, and this contradicts the assumption on s .

Conversely, if $\sup J_s > b_s$, there exists $\varepsilon > 0$ such that $\begin{pmatrix} a_s \\ b_s + \varepsilon \end{pmatrix} \in K_s$. Then we can write

$$\begin{pmatrix} a_s \\ b_s + \gamma \end{pmatrix} = \sum_{t \in T} \lambda_t^1 \begin{pmatrix} a_t \\ b_t \end{pmatrix},$$

with $\lambda^1 \in \mathbb{R}_+^{(T)}$, $\gamma \geq \varepsilon > 0$ and $s \notin \text{supp } \lambda^1$.

Let us suppose that s is not strongly extraneous. Then, there exists $c \in \mathbb{R}^n$ and $\lambda^2 \in \Lambda^*(c)$ such that $s \in \text{supp } \lambda^2$. Hence,

$$\begin{pmatrix} c \\ v^D(c) \end{pmatrix} = \sum_{t \in T} \lambda_t^2 \begin{pmatrix} a_t \\ b_t \end{pmatrix} \quad \text{with } \lambda_s^2 > 0.$$

Defining

$$\lambda_t^3 := \begin{cases} \lambda_t^2 + \lambda_s^2 \lambda_t^1, & \text{if } t \neq s, \\ 0, & \text{if } t = s, \end{cases}$$

we have

$$\sum_{t \in T} \lambda_t^3 a_t = \sum_{t \in T \setminus \{s\}} \lambda_t^2 a_t + \lambda_s^2 \sum_{t \in T \setminus \{s\}} \lambda_t^1 a_t = c - \lambda_s^2 a_s + \lambda_s^2 a_s = c$$

and

$$\begin{aligned} \sum_{t \in T} \lambda_t^3 b_t &= \sum_{t \in T \setminus \{s\}} \lambda_t^2 b_t + \lambda_s^2 \sum_{t \in T \setminus \{s\}} \lambda_t^1 b_t \\ &= v^D(c) - \lambda_s^2 b_s + \lambda_s^2 (b_s + \gamma) = v^D(c) + \lambda_s^2 \gamma > v^D(c), \end{aligned}$$

in contradiction with $\lambda^2 \in \Lambda^*(c)$. So, s is strongly extraneous.

Finally, observe that $\sup J_s > b_s$ entails the existence of $\varepsilon > 0$ such that $\begin{pmatrix} a_s \\ b_s + \varepsilon \end{pmatrix} \in K_s$. Since we are assuming that σ is consistent, s is strongly redundant. ■

Proposition 5.3 *If s is extraneous (strongly extraneous) and σ is consistent (σ is not strongly inconsistent, respectively), then s is DVS.*

Proof. First we assume that s is extraneous and σ is consistent.

Since $a_s \in M_s$ (by Corollary 5.2), $-\infty < \sup I_s < +\infty$ and so I_s is a halfline.

Two cases can arise:

Case 1: I_s is closed. Then, by Proposition 5.1, $I_s = J_s$.

Case 2: I_s is open. Let $\bar{\alpha} := \sup I_s$ and $\bar{\beta} := \sup J_s$. We shall prove that $\bar{\alpha} = \bar{\beta}$.

Since $b_s \in I_s$ and I_s is open, $b_s < \bar{\alpha}$. Let $\varepsilon > 0$ arbitrarily such that $b_s + \varepsilon < \bar{\alpha}$, i.e., $b_s + \varepsilon \in I_s$. Then we can write

$$\begin{pmatrix} a_s \\ b_s + \varepsilon \end{pmatrix} = \sum_{t \in T \setminus \{s\}} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \lambda_s \begin{pmatrix} a_s \\ b_s \end{pmatrix} + \mu \begin{pmatrix} 0_n \\ -1 \end{pmatrix}, \quad (5)$$

with $\lambda \in \mathbb{R}_+^{(T)}$ and $\mu \geq 0$.

If $\lambda_s \geq 1$, then

$$\begin{pmatrix} 0_n \\ \varepsilon \end{pmatrix} = \sum_{t \in T \setminus \{s\}} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + (\lambda_s - 1) \begin{pmatrix} a_s \\ b_s \end{pmatrix} + \mu \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in K,$$

contradicting the assumption on σ .

Alternatively, if $0 \leq \lambda_s < 1$, from (5) we obtain

$$(1 - \lambda_s) \begin{pmatrix} a_s \\ b_s + \varepsilon \end{pmatrix} = \sum_{t \in T \setminus \{s\}} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \begin{pmatrix} 0_n \\ -\mu - \lambda_s \varepsilon \end{pmatrix} \in K_s,$$

so that, $\begin{pmatrix} a_s \\ b_s + \varepsilon \end{pmatrix} \in K_s$ and $b_s + \varepsilon \leq \bar{\beta}$. Hence, $\bar{\alpha} = \bar{\beta}$.

Since $\sup I_s = \sup J_s$ in both cases, s is DVS according to Proposition 4.1.

Now we suppose that s is strongly extraneous and σ is not strongly inconsistent.

By Proposition 5.2, $\sup J_s > b_s$. Moreover, since σ is not strongly inconsistent, I_s and J_s can not be lines and reasoning as in the first part of the proof, we conclude that s is DVS. ■

Proposition 5.4 *If σ is inconsistent then any $s \in T$ is nonessential. Otherwise, s is nonessential if and only if $(\text{extr } F) \cap H_s = \emptyset$.*

Proof. The first statement is trivial. So we assume that σ is consistent.

Assume that $(\text{extr } F) \cap H_s \neq \emptyset$. Let $\bar{x} \in (\text{extr } F) \cap H_s$. Then $\bar{x} \in (\text{extr } F^*(a_s)) \cap H_s$ and so s is essential.

Conversely assume that s is essential. Let $c \in \mathbb{R}^n$ such that $(\text{extr } F^*(c)) \cap H_s \neq \emptyset$. If $c = 0_n$, then $F^*(c) = F$ and $(\text{extr } F) \cap H_s \neq \emptyset$. So we can assume that $c \neq 0_n$.

Let $\bar{x} \in (\text{extr } F^*(c)) \cap H_s$. Since $H := \{x \in \mathbb{R}^n \mid c'x = c'\bar{x}\}$ is a supporting hyperplane to F at \bar{x} , we have

$$(\text{extr } F) \cap H = \text{extr } (F \cap H) = \text{extr } F^*(c).$$

Hence $\bar{x} \in (\text{extr } F^*(c)) \cap H_s \subset (\text{extr } F) \cap H_s$ and $(\text{extr } F) \cap H_s \neq \emptyset$. ■

Proposition 5.5 *Let σ be consistent and $s \in T$. Then the following statements hold:*

- (i) *If s is strongly redundant, then it is strongly extraneous.*
- (ii) *If s is strongly extraneous, then it is nonbinding.*
- (iii) *If s is nonbinding, then it is nonessential.*
- (iv) *If s is nonessential and $\text{extr } F \neq \emptyset$, then it is nonbinding.*

Proof. (i) Since $\sigma_s \cup \{a'_s x = b_s\}$ is strongly inconsistent there exist $\lambda \in \mathbb{R}_+^{(T \setminus \{s\})}$, $\alpha \in \mathbb{R}$ and $\mu \in \mathbb{R}_+$ such that

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} = \sum_{t \in T \setminus \{s\}} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \alpha \begin{pmatrix} a_s \\ b_s \end{pmatrix} + \mu \begin{pmatrix} 0_n \\ -1 \end{pmatrix}. \quad (6)$$

If $\alpha \geq 0$, then $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in K$, contradicting the assumption on σ .

Since $\alpha < 0$, multiplying by $|\alpha|^{-1}$ both members of (6), we get

$$\begin{pmatrix} a_s \\ b_s + |\alpha|^{-1}(1 + \mu) \end{pmatrix} = \sum_{t \in T \setminus \{s\}} |\alpha|^{-1} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} \in K_s,$$

so that $\sup J_s \geq b_s + |\alpha|^{-1}(1 + \mu) > b_s$ and the conclusion follows from Proposition 5.2.

(ii) Again by Proposition 5.2, we have $\sup J_s > b_s$. Let $\varepsilon > 0$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that

$$\begin{pmatrix} a_s \\ b_s + \varepsilon \end{pmatrix} = \sum_{t \in T \setminus \{s\}} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \lambda_s \begin{pmatrix} 0_n \\ -1 \end{pmatrix}. \quad (7)$$

Let $x \in F$. Multiplying by $(x', -1)$ both members of (7), we get $a'_s x - (b_s + \varepsilon) \geq 0$.

Then $a'_s x > b_s$ and so $x \notin H_s$. Hence $F \cap H_s = \emptyset$.

(iii) If $F \cap H_s = \emptyset$, then $(\text{extr } F) \cap H_s = \emptyset$ and so s is nonessential by Proposition 5.4.

(iv) We assume that $\text{extr } F \neq \emptyset$ and s is binding.

Consider an arbitrary $\bar{x} \in F \cap H_s$. Since $F = (\text{conv extr } F) + 0^+ F$, there exist $p \in \mathbb{N}$, $\{x^1, \dots, x^p\} \subset \text{extr } F$, $\{\lambda_1, \dots, \lambda_p\} \subset \mathbb{R}_+$, and $y \in 0^+ F$ such that $\sum_{i=1}^p \lambda_i = 1$ and $\bar{x} = \sum_{i=1}^p \lambda_i x^i + y$. Since $\bar{x} \in H_s$, we have

$$b_s = a'_s \bar{x} = \sum_{i=1}^p \lambda_i a'_s x^i + a'_s y \geq \sum_{i=1}^p \lambda_i b_s + 0 = b_s,$$

and this entails $a'_s x^i = b_s$ for all $i \in \{1, \dots, p\}$. Hence, $\{x^1, \dots, x^p\} \subset (\text{extr } F) \cap H_s$, so that s is essential. ■

The assumption $\text{extr } F \neq \emptyset$ cannot be eliminated in (iv): replace -1 in the system of Example 1.1 with 1.

6 Superfluous variables relative to the dual optimal set

Example 6.1 Let $\sigma = \{x_1 \geq 1; x_1 \geq 1\}$ in \mathbb{R}^2 and $s = 2$. Obviously s is superfluous relative to the primal mappings, DVS (by Proposition 4.1) and extraneous (by Corollary 5.1). Nevertheless it is neither strongly extraneous nor DOS: observe that $\Lambda_2^*(c) \neq \Lambda^*(c)$ for all $c \in M \setminus \{0_2\} = \text{cone } \{(1, 0)'\} \setminus \{0_2\}$ since

$$\Lambda_2^*(c) = \begin{cases} \{\mu\}, & \text{if } c = \mu(1, 0)', \mu \geq 0, \\ \emptyset, & \text{if } c \notin M, \end{cases}$$

and

$$\Lambda^*(c) = \begin{cases} \text{conv} \{(\mu, 0)', (0, \mu)'\}, & \text{if } c = \mu(1, 0)', \mu \geq 0, \\ \emptyset, & \text{if } c \notin M. \end{cases}$$

In fact, the last two concepts are basically equivalent, as the next result shows.

Proposition 6.1 *Given $s \in T$ the following statements hold:*

- (i) *If σ is strongly inconsistent, then s is DOS if and only if $D_s(c)$ is unsolvable for all $c \in \mathbb{R}^n$.*
- (ii) *If s is DOS, then s is strongly extraneous, and the converse is true if σ is not strongly inconsistent.*

Proof. (i) We assume that σ is strongly inconsistent. In such a case $\Lambda^*(c) = \emptyset$ for all $c \in \mathbb{R}^n$, so that s is DOS if and only if $\Lambda_s^*(c) = \emptyset$ for all $c \in \mathbb{R}^n$.

(ii) Suppose that s is DOS. If $\Lambda^*(c) \neq \emptyset$, taking $\lambda \in \Lambda^*(c) = \overline{\Lambda_s^*(c)}$ we have $\lambda_s = 0$, i.e., $s \notin \text{supp } \lambda$. Hence, s is strongly extraneous.

Finally we suppose that s is strongly extraneous and σ is not strongly inconsistent. By Proposition 5.3, s is DVS and by Proposition 4.1, $a_s \in M_s$ (i.e., $M = M_s$) and $\sup J_s = \sup I_s$ (i.e., $v_s^D(c) = v^D(c)$). Let us show that $\overline{\Lambda_s^*(c)} = \Lambda^*(c)$ for every $c \in \mathbb{R}^n$. If $c \notin M = M_s$, $\overline{\Lambda_s^*(c)} = \Lambda^*(c) = \emptyset$. Otherwise, $v^D(c) = v_s^D(c) = \sup J_s > b_s$ by Proposition 5.2 and two cases can arise:

Case 1: $v^D(c) = v_s^D(c) = +\infty$ and so $\overline{\Lambda_s^*(c)} = \Lambda^*(c) = \emptyset$.

Case 2: $b_s < v^D(c) = v_s^D(c) < +\infty$. If there exists $c \in M$ and $\lambda \in \Lambda^*(c)$ such that $\lambda \notin \overline{\Lambda_s^*(c)}$, then $\lambda_s > 0$ and so $s \in \text{supp } \lambda$, contradicting the assumption on s . Thus $\Lambda^*(c) \subset \overline{\Lambda_s^*(c)}$ whereas the opposite inclusion trivially holds in this case.

Hence, $\overline{\Lambda_s^*(c)} = \Lambda^*(c)$ for every $c \in \mathbb{R}^n$ and s is DOS. ■

The next example shows that the equivalence in (ii) fails if σ is strongly inconsistent. It also shows that we can have $\Lambda^*(c) = \overline{\Lambda_s^*(c)}$, for all $c \in \mathbb{R}^n \setminus \{0_n\}$, and nevertheless $\Lambda^*(0_n) \neq \overline{\Lambda_s^*(0_n)}$.

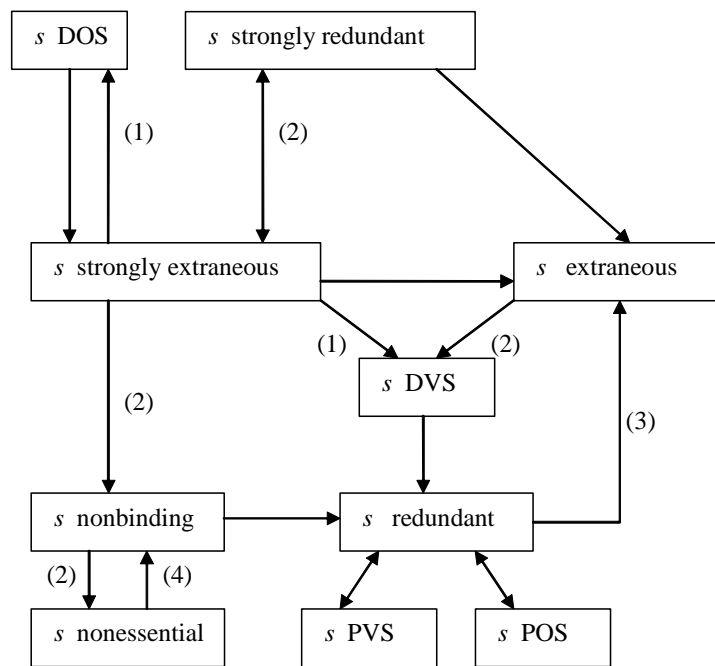
Example 6.2 Let us consider $\sigma = \{x_2 \geq t, t \in]0, 1[; -x_2 \geq 0\}$ in \mathbb{R}^2 and let us associate to the constraint $-x_2 \geq 0$ the index $s = 1$. It can be realized that σ is strongly inconsistent (and so, all constraints are strongly extraneous and inessential), whereas $\Lambda^*(c) = \overline{\Lambda_1^*(c)} = \emptyset$, for all $c \in \mathbb{R}^n \setminus \{0_2\}$, $\Lambda^*(0_2) = \emptyset$ and $\Lambda_1^*(0_2)$ is a singleton set formed by the null function. Thus $s = 1$ is not DOS (in fact, it is not superfluous in any sense).

7 Conclusion

We have characterized in a geometric way all the phenomena of excess information in parametric linear optimization introduced in Section 1, so that most of these properties can be checked in practice (observe that $\sup J_s$ and $\sup I_s$ can be computed by solving suitable LP or LSIP problems). Moreover, these characterizations allowed us to prove the relationships summarized in Figure 1. Examples can be given showing that any existing relationship between these phenomena can be derived from this diagram.

If $|T| < \infty$ and σ is consistent, then there exist two clusters of equivalent properties:

- (A) redundant, PVS, POS, DVS, and extraneous.
- (B) strongly redundant, DOS, strongly extraneous, nonbinding, and nonessential (provided F does not contain lines).



- (1) If σ is not strongly inconsistent.
- (2) If σ is consistent.
- (3) If σ_s is FM (e.g., $|T| < \infty$).
- (4) If $F \neq \emptyset$ does not contain lines.

Figure 1

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